1.2: Integrals as General and Particular Solutions

Consider a first-order equation $dy/dx = f(x, y)$. The solution to this equation becomes simple if the function on the right-hand side f does not depend on the dependent variable y. In this case,

$$\frac{dy}{dx} = f(x) \tag{1}$$

has the simple solution

$$y(x) = \int f(x)dx + C. \tag{2}$$

Definition 1.

(a) (2) is called a **general solution** of (1), meaning that it involves an arbitrary constant (C in this case). The graphs of two such solutions for different C are easily seen to be parallel.

(b) If we were to add an initial condition $y(x_0) = y_0$ to (1) then we would be able to solve for the constant C and would obtain what is called the **particular solution.**

Example 1. Solve the initial value problem

$$\frac{dy}{dx} = 2x + 3, \quad y(1) = 2.$$

$y = \int 2x+3dx = x^2+3x+C$

$2 = y(1) = 1^2+3\cdot 1+C \Rightarrow C = -2$. So

$y = x^2+3x-2$

Position, Velocity and Acceleration

One of the most important applications for differential equations in the simple case mentioned above is describing motion. In this instance, we start with a **position** function $x = f(t)$. Then the **velocity** function is given by

$$v(t) = f'(t) \quad \text{or} \quad v(t) = \frac{dx}{dt}. \tag{3}$$

Its **acceleration** function is given by

$$a(t) = v'(t) = f''(t) \quad \text{or} \quad a(t) = \frac{dv}{dt} = \frac{d^2f}{dt^2}. \tag{4}$$

Thus the motion of a particle can often be described as a second-order equation.

Example 2. A lunar lander is falling freely toward the surface of the moon at a speed of 450 meters per second. Its retrorockets, when fired, provide a constant deceleration of 2.5 meters per second per second (this includes the gravitational pull of the moon). At what height above the lunar surface should the retrorockets be activated to ensure a "soft touchdown;" i.e. $v = 0$ at impact? $V_0 = -450$ $\quad a(t) = 2.5 \Rightarrow v(t) = 2.5t - 450$

position $x(t) = \int v(t)\,dt = 1.25t^2 - 450t + C$

$v(t) = 0$ when $t = 180s$. So $x(180) = 0 \Rightarrow C = 40,500$

Definition 2. The **weight** W of a body is the force exerted on the body by gravity. Hence, on the Earth, if something has a mass of m kg then its weight is

$$W = (mass) \cdot (gravity) = (m \text{ kg}) \cdot (9.8 \text{ m/s}^2) = 9.8 \cdot m \text{ Newtons.}$$

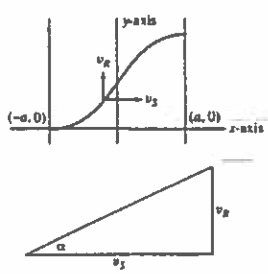
Exercise 1. Suppose that a ball is thrown straight upward from the ground ($y_0 = 0$) with initial velocity $v_0 = 96$ ft/s. Find the time at which the ball reaches its maximum height and determine its height at that time.

~~gravity = v_0 = 96 ft/s²~~ gravity = 32 f/s² so

$v(t) = -32t + 96$. $v(t) = 0$ when $t = 3s$.

Clearly a max, so $y(3) = -16(3)^2 + 96(3) = 144$ ft.

Example 3.



Suppose the river has a velocity of $v_R = v_0(1 - \frac{x^2}{a^2})$ and a swimmer starts at point $(-a, 0)$ and swims due East with a constant velocity of v_S. Hence the swimmer's direction is given by

$$\tan\alpha = \frac{v_R}{v_S} = \frac{dy}{dx}.$$

Assuming that the river is 1 mile wide, its midstream velocity is $v_0 = 9$ mi/h and the swimmer's velocity is $v_S = 3$ mi/h, find how far downstream the swimmer will hit the opposite bank.

$V_R = 9(1 - 4x^2)$. So $\dfrac{dy}{dx} = \dfrac{9(1-4x^2)}{3} = 3(1 - 4x^2)$. Thus $y(x) = 3x - 4x^3 + C$

$a = \frac{1}{2}$ $\qquad$ But $y(-\frac{1}{2}) = 0$, thus $C = 1$

Homework. 1-17, 25-33 (odd) $\qquad$ So $y(\frac{1}{2}) = 2$ miles downstr